

Concise formulary

Some definitions and formulae from the [main text](#) are represented in concise form¹.

1 Q0-triangle

Let be j, m, n natural numbers, k an integer number with $|(n+k)/2|$ smaller or equal n , i.e. k is contained in the interval $[-n, n]$, starting with $-n$, stepping by 2 until $+n$. Furthermore p is a number contained in the interval $[0, 1]$. We define

$$Q0P(n, k, p) := \frac{p^{(n+k)/2} (1-p)^{(n-k)/2} n!}{\left(\frac{n+k}{2}\right)! \left(\frac{n-k}{2}\right)!}$$

The function $Q0P(n, k, p)$ represents the probability of reaching coordinate k after n steps of a Bernoulli random walk, if the probability of a step in positive k -direction (e.g. right hand direction) is equal to p (and so for a step to the opposite direction is equal to $1-p$). The numbers $Q0(n, k)$ of the Q0-triangle correspond to the special case of same probabilities for steps to the right and to the left, i.e. for $p=(1-p)=0.5$:

$$Q0(n, k) = Q0P(n, k, 0.5)$$

The probabilities $Q0Z(n)$ for return ("central meeting probabilities") correspond to the special case $k = 0$:

$$Q0Z(n) = Q0P(n, 0, 0.5) = \frac{n!}{2^n \left(\frac{n}{2}\right)! \left(\frac{n}{2}\right)!}$$

2 Q1-triangle

The Q1-triangle results from a superposition of two Q0-triangles with opposite sign, starting in position $n=1, k=\pm 1$ after multiplication by $1/2$. Addition of both means a "discrete differentiation" along k .

$$Q1(n, k) = -\frac{k}{n} Q0(n, k)$$

The absolute values $|Q1(n, k)|$ also arise, if after starting in row $n=1$ the following rows are constructed in usual way, but the numbers in the row centers $k=0$ are set to 0 in every row with even row number respectively, are so to speak "flown out", so that they can't be sources subsequently. Let for every even row number n be $-Q2Z(n)$ "flowing out probability" there, i.e. the probability for flowing out centrally. $Q2Z(n)$ is equivalent to the 1nd (discrete) derivative of $Q1(n, k)$ in $k=0$ along k , i.e. $Q2Z(n) = (Q1(n-1, 1) - Q1(n-1, -1))/2$; so $Q2Z(n)$ is in $k=0$ the 2nd derivative of $Q0(n, k)$ along k . It holds:

¹ In wqm (contained in the download of the older texts) is a more extensive formulary.

$$Q2Z(n) = \frac{Q1(n-1,1) - Q1(n-1,-1)}{2} = -\frac{n!}{2^n(n-1) \left(\frac{n}{2}\right)!} = -\frac{Q0Z(n)}{n-1}$$

3 Q0M-triangle

$$Q0M(n,k) := \frac{(-0.5)^{(n+k)/2} 0.5^{(n-k)/2} n!}{\left(\frac{n+k}{2}\right)! \left(\frac{n-k}{2}\right)!} = (-1)^{(n+k)/2} Q0(n,k)$$

Q0M(n,k) is in case of odd n antisymmetric and in case of even n symmetric regarding to k=0 .
So addition behavior of right and left side is like the one of amplitudes of fermions resp. bosons.

4 Taylor series expansions

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} Q0P\left(2n,0, \frac{1+\sqrt{1-x^2}}{2}\right) = \sum_{n=0}^{\infty} Q0Z(2n)x^{2n}$$

$$\sqrt{1-x^2} = \sum_{n=0}^{\infty} Q2Z(2n)x^{2n}$$

$$\frac{1}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty} Q0M(2n,0)x^{2n}$$

5 Limits

$$\lim_{n \rightarrow \infty} \left[\frac{\sum_{l=0}^n \sum_{m=0}^{l/2} Q0Z(2m)}{\sqrt{\frac{8n^3}{9\pi}}}, \frac{\sum_{m=0}^{n/2} Q0Z(2m)}{\sqrt{\frac{2n}{\pi}}}, \frac{Q0Z(n)}{\sqrt{\frac{2}{\pi n}}}, \frac{-Q2Z(n)}{\sqrt{\frac{2}{\pi n^3}}} \right] = [1, 1, 1, 1]$$

6 Multiple discrete differentiation (Formation of higher-order finite differences)

Similarly to the analytical case multiple discrete differentiation can be defined recursively (by formation of higher-order finite differences). Let be QDP(d,n,k,p) the d times along k differentiated function Q0P (n, k, p), then

$$QDP(0,n,k,p) = Q0P(n,k,p)$$

and for $n \geq d \geq 1$

$$QDP(d,n,k,p) = \frac{QDP(d-1,n-1,k+1,p) - QDP(d-1,n-1,k-1,p)}{2}$$

At this $n \geq d$ is necessary that enough values are available to build a (finite) difference of d-th order. Let $DF(d,n,k) := QDP(d,n,k,1/2) / Q0P(n,k,1/2)$; We get

$$DF(1,n,k) = -\frac{k}{n}$$

$$DF(2, n, k) = \frac{k^2 - n}{n(n - 1)}$$

$$DF(3, n, k) = \frac{-k^3 + 3kn - 2k}{n(n - 1)(n - 2)}$$

$$DF(4, n, k) = \frac{k^4 - 6k^2n + 8k^2 + 3n^2 - 6n}{n(n - 1)(n - 2)(n - 3)}$$

$$DF(5, n, k) = \frac{-k^5 + 10k^3n - 20k^3 - 15kn^2 + 50kn - 24k}{n(n - 1)(n - 2)(n - 3)(n - 4)}$$

$$DF(6, n, k) = \frac{k^6 - 15k^4n + 40k^4 + 45k^2n^2 - 210k^2n + 184k^2 - 15n^3 + 90n^2 - 120n}{n(n - 1)(n - 2)(n - 3)(n - 4)(n - 5)}$$

6.1 Binomial coefficients and multiple differentiation (example matrix)

[{BinCoeffDiffMatrix}](#) The representation of operators as matrices is often useful in discrete considerations. Here a matrix representation of the operator for discrete differentiation in form of an example matrix with high number of dimensions for clarification of the development: Multiplication of a 15-dimensional vector by the following matrix

$$\Delta := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} * 1/2$$

means first order discrete differentiation "along" the index k of the vector components (calculation of the finite first order difference - the shift dk of the index k is 2, therefore division by 2). Multiplication by the n-th power Δ^n of this matrix yields n-th order discrete differentiation (formation of the finite n-th order difference). For example means multiplication by

$$\Delta^6 = \begin{pmatrix} -20 & 0 & 15 & 0 & -6 & 0 & 1 & 0 & 0 & 1 & 0 & -6 & 0 & 15 & 0 \\ 0 & -20 & 0 & 15 & 0 & -6 & 0 & 1 & 0 & 0 & 1 & 0 & -6 & 0 & 15 \\ 15 & 0 & -20 & 0 & 15 & 0 & -6 & 0 & 1 & 0 & 0 & 1 & 0 & -6 & 0 \\ 0 & 15 & 0 & -20 & 0 & 15 & 0 & -6 & 0 & 1 & 0 & 0 & 1 & 0 & -6 \\ -6 & 0 & 15 & 0 & -20 & 0 & 15 & 0 & -6 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -6 & 0 & 15 & 0 & -20 & 0 & 15 & 0 & -6 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & -6 & 0 & 15 & 0 & -20 & 0 & 15 & 0 & -6 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -6 & 0 & 15 & 0 & -20 & 0 & 15 & 0 & -6 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -6 & 0 & 15 & 0 & -20 & 0 & 15 & 0 & -6 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & -6 & 0 & 15 & 0 & -20 & 0 & 15 & 0 & -6 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -6 & 0 & 15 & 0 & -20 & 0 & 15 & 0 & -6 \\ -6 & 0 & 1 & 0 & 0 & 1 & 0 & -6 & 0 & 15 & 0 & -20 & 0 & 15 & 0 \\ 0 & -6 & 0 & 1 & 0 & 0 & 1 & 0 & -6 & 0 & 15 & 0 & -20 & 0 & 15 \\ 15 & 0 & -6 & 0 & 1 & 0 & 0 & 1 & 0 & -6 & 0 & 15 & 0 & -20 & 0 \\ 0 & 15 & 0 & -6 & 0 & 1 & 0 & 0 & 1 & 0 & -6 & 0 & 15 & 0 & -20 \end{pmatrix} * 1/64$$

6-th order discrete Differentiation resp. calculation of the finite 6-th order difference. The rows resp. columns of the matrix Δ^n contain the binomial coefficients, divided by 2^n , in this example the numbers $6!/(k! \cdot (6 - k)! \cdot 2^6) = Q_0(6, 2k-6)$.

7 Special differences

horizontal (along localization):

$$Q0(n, k+2) - Q0(n, k) = 2 QDP(1, n+1, k+1, \frac{1}{2}) = 2 Q1(n+1, k+1) = \frac{-2(k+1)}{n+1} Q0(n+1, k+1)$$

vertical (along time):

$$Q0(n+2, k) - Q0(n, k) = \frac{k^2 - n - 2}{(n+2)(n+1)} Q0(n+2, k)$$

Correspondence in the middle:

$$Q1(n+1, 1) = Q0(n+2, 0) - Q0(n, 0) = \frac{1}{2} (Q0(n, 2) - Q0(n, 0))$$

7.1 Schrödinger discretely

$$Q0(n, k-2) - 2 Q0(n, k) + Q0(n, k+2) = 4 QDP(2, n+2, k, \frac{1}{2}) = 4 (Q0(n+2, k) - Q0(n, k))$$

$$Q1(n, k-2) - 2 Q1(n, k) + Q1(n, k+2) = 4 QDP(3, n+2, k, \frac{1}{2}) = 4 (Q1(n+2, k) - Q1(n, k))$$

Remark: Relation of the discrete second derivative of Q0(n-2,k) to Q0(n,k):

$$Q0(n-2, k-2) - 2 Q0(n-2, k) + Q0(n-2, k+2) = Q0(n, k) \frac{4(k^2 - n)}{n(n-1)}$$

8 Scalar products

8.1 horizontal

$$\sum_{k=-m/2}^{m/2} Q0(m, 2k) Q0(n, 2k+j) = Q0(m+n, j)$$

$$\sum_{k=-n/2}^{n/2} Q0(n, 2k)^2 = Q0(2n, 0)$$

$$\sum_{k=-m/2}^{m/2} Q1(m, 2k) Q1(n, 2k) = Q1(m+n-1, -1) = -Q2Z(m+n)$$

$$2 \sum_{k=-n/2}^{n/2} (2k)^2 Q0(n, 2k)^2 - 0.5 \sum_{k=-(n+1)/2}^{(n-1)/2} Q0(n-1, 2k)^2 = n Q0(2n-2, 0)$$

8.2 vertical

$$\sum_{n=1}^{2j-1} \sum_{k=-n/2}^{n/2} Q1(n, 2k) Q1(2j-n, 2k) = -(2j-1) Q2Z(2j) = Q0Z(2j) \quad \text{\textit{[skahove]}}$$

$$\sum_{k=0}^n Q2Z(2k) Q2Z(2n-2k) = \left(\sum_{k=1}^n Q2Z(2k) Q2Z(2n-2k) \right) + Q2Z(2n) = 0 .$$

$$\sum_{k=0}^n \left(\sum_{j=0}^k Q2Z(2j) \right) \left(\sum_{j=0}^{n-k} Q2Z(2j) \right) = \sum_{k=0}^n Q0Z(2k) Q0Z(2n-2k) = 1 .$$

8.3 Orthogonality after multiple discrete differentiation (analogously to Hermite polynomials)

{HermPolDiscrete}

Let be $d, l \geq n$.

We define the weighted scalar product QSP by

$$QSP(d, l, n, p) := \sum_{k=-n/2}^{n/2} \frac{1}{Q0P(n, 2k, p)} QDP(d, n, 2k, p) QDP(l, n, 2k, p).$$

Then for $d \neq l$ is valid: $QSP(d, l, n, p) = 0$ (i.e. orthogonality) and otherwise

$$QSP(d, d, n, p) = \frac{1}{2^n (4p(1-p))^d Q0(n, n-2d)} = \frac{1}{p^{2d-n} 4^d Q0P(n, n-2d, p)}$$

particularly

$$QSP\left(d, d, n, \frac{1}{2}\right) = \frac{1}{2^n Q0(n, n-2d)}$$

The denominator in the last expression corresponds to the number of the way possibilities from point (0,0) to point (n,n-2d) in the Q0-triangle.

9 Sums

$$Q0Z(2n) = \sum_{m=0}^n Q2Z(2m) = 1 + \sum_{m=1}^n Q2Z(2m)$$

10 Moments

10.1 vertical

$$\frac{\sum_{m=0}^n Q0Z(2m)}{(2n+1)Q0Z(2n)} = \frac{\sum_{m=0}^{n-1} Q0Z(2m)}{2nQ0Z(2n)} = \frac{-Q0Z(2n)}{(2n-1)Q2Z(2n)} = \frac{3\sum_{m=0}^n 2m Q0Z(2m)}{2n(2n+1)Q0Z(2n)} = 1$$

10.2 horizontal

$$-\sum_{k=-n}^n Q1(2n, 2k) 2k = -\sum_{k=-n}^{n+1} Q1(2n+1, 2k-1) (2k-1) = 1$$

$$\sum_{k=0}^n 2k Q0(2n, 2k) = n Q0Z(2n)$$

$$2\sum_{k=0}^{n/2} (2k)^2 Q0(n, 2k) = 4\sum_{k=0}^{n/4} (4k)^2 Q0(n, 4k) = n$$

$$\sum_{k=-n}^0 (2k)^2 Q1(2n, 2k) = 2n Q0Z(n)$$

$$\sum_{k=-n}^0 (2k)^3 Q1(2n, 2k) = 1 - 3n$$

10.2.1 relative to the border

$$\sum_{k=-n/2}^{n/2} (2k+n)^2 Q0(n,2k) = n(n+1)$$

$$\sum_{k=-n/2}^{n/2} (2k-n)^2 Q1(n,2k) = 2n$$

11 Sums and moments for variable p

This chapter contains some elementary formulae for variable p (and n>0).

The first finite difference (discrete derivative) of Q0P is

$$Q1P(n, k, p) := \frac{Q0P(n-1, k+1, p) - Q0P(n-1, k-1, p)}{2}$$

11.1 Sums

$$\sum_{k=-n/2}^{n/2} Q0P(n, 2k, p) = 1$$

$$\sum_{k=-n/2}^{n/2} Q1P(n, 2k, p) = 0$$

11.2 Deviation relative to the border

In the border the probabilities p and 1-p are very different. With p->0 also v->0 (low temperature).

$$\sum_{k=-n/2}^{n/2} (n+2k) Q0P(n, 2k, p) = 2np$$

$$\sum_{k=-n/2}^{n/2} (-n-2k) Q1P(n, 2k, p) = 1$$

11.3 Deviation relative to the origin

In the center the probabilities p and 1-p are nearly equal, i.e. p-> 1/2 and with that v-> C (the borderline case p=1/2 resp. v=C is represented by Q0 and Q1).

$$\sum_{k=-n/2}^{n/2} (2k) Q0P(n, 2k, p) = 2np - n = n(2p - 1)$$

$$\sum_{k=-n/2}^{n/2} (-2k) Q1P(n, 2k, p) = 1$$

12 Analytic representations

Let be

$$Q0E(n, k) := \sqrt{\frac{2}{\pi n}} e^{-k^2/(2n)}$$

$$Q1E(n, k) := -\frac{k}{n} Q0E(n, k)$$

i.e.

$$Q1E(n, k) = \frac{\partial}{\partial k} Q0E(n, k)$$

then is valid for every sequence (k_n) with $(k_n)^3/n^2 \rightarrow 0$ für $n \rightarrow \infty$ (p. 80 [\[likr\]](#))

$$\lim_{n \rightarrow \infty} \left[\frac{Q0E(n, k_n)}{Q0(n, k_n)}, \frac{Q1E(n, k_n)}{Q1(n, k_n)} \right] = [1, 1]$$

Again tested approximation: For $|k/n| \ll 1$ (for example up to $n=100$ for $|k/n| < 0.2$) the function $Q0(n, k) = Q0P(n, k, 0.5)$ is related to the normal distribution:

$$Q0(n, k) / Q0E(n, k) = Q0(n, k) / \left(\left(\sqrt{\frac{2}{n\pi}} \right) e^{-\left(\frac{k^2}{2n}\right)} \right) \rightarrow 1$$

for larger $|k/n|$ this quotient is first decreasing, then invalid.

12.1 Schrödinger analytically

$$\frac{\partial^2}{\partial k^2} Q0E(n, k) = 2 \frac{\partial}{\partial n} Q0E(n, k) = \frac{k^2 - n}{n^2} Q0E(n, k)$$

$$\frac{\partial^2}{\partial k^2} Q1E(n, k) = 2 \frac{\partial}{\partial n} Q1E(n, k) = \frac{k^2 - 3n}{n^2} Q1E(n, k)$$

$$\left[\frac{\partial}{\partial k} \right]^{2j} Q0E(n, k) = 2^j \left[\frac{\partial}{\partial n} \right]^j Q0E(n, k)$$

$$\left[\frac{\partial}{\partial k} \right]^{2j} Q1E(n, k) = 2^j \left[\frac{\partial}{\partial n} \right]^j Q1E(n, k)$$

12.2 Behavior for $n \rightarrow \infty$; Dirac delta-function

[{DiracDeltaFu}](#) It is

$$\int_{-\infty}^{\infty} \frac{Q0E(n, k)}{2} dk = 1$$

The behavior for $n \rightarrow \infty$ can be illustrated by a n proportional scale fitting, i.e. by a horizontal compression and a vertical stretching by respectively the factor n . This doesn't touch the value of the integral:

$$\int_{-\infty}^{\infty} \frac{n Q0E(n, nk)}{2} dk = \int_{-\infty}^{\infty} \sqrt{\frac{n}{2\pi}} e^{-k^2 n/2} dk = 1$$

For $n \rightarrow \infty$ therefore the function $f(x) = n Q0E(n, nx) / 2$ converges towards the Dirac delta-function.

12.3 Multiple differentiation and Hermite polynomials

[{HermPol}](#) The Hermite polynomials $H_n(x)$ are (except sign) special cases of pre-factors resulting from multiple differentiation:

$$H_n(x) = (-1)^n \frac{\left[\frac{\partial}{\partial x} \right]^n Q0E\left(\frac{1}{2}, x\right)}{Q0E\left(\frac{1}{2}, x\right)} = (-1)^n \frac{\left[\frac{\partial}{\partial x} \right]^n Q0E(100, 10x\sqrt{2})}{Q0E(100, 10x\sqrt{2})}$$